Computer Science Department

TECHNICAL REPORT

"On Two Conjectures Regarding Eigenvalue Perturbations and a Common Counterexample"

by

James Weldon Demmel

Technical Report #220
May, 1986

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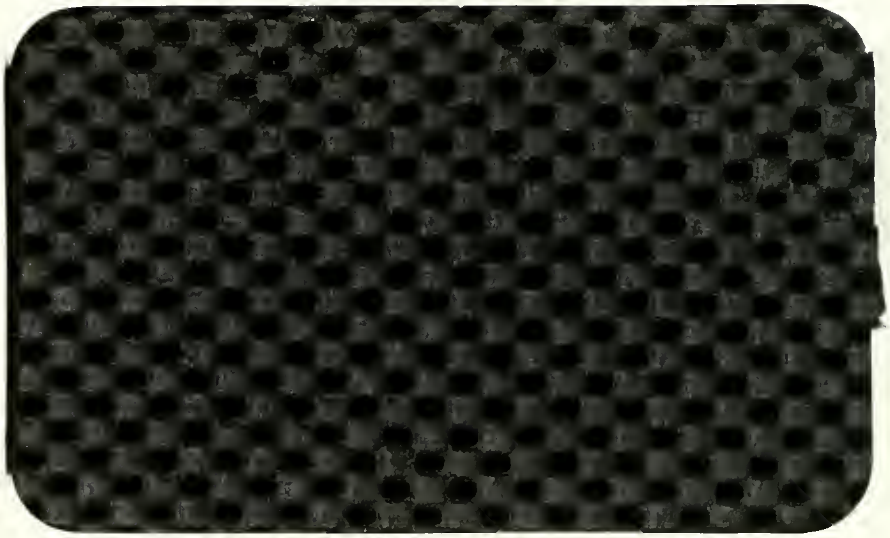

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Abstract

Recently Van Loan and Demmel made conjectures about eigenvalue perturbations. Van Loan's conjecture concerned the smallest perturbation that makes a stable matrix unstable, and Demmel's concerned the smallest perturbations that makes two matrices with disjoint spectra have a common eigenvalue. We show that the truth of either of these conjectures would imply the truth of a third weaker conjecture for which we supply a counterexample.

Recently Van Loan and Demmel made reasonable sounding conjectures about eigenvalue perturbations. Van Loan's conjecture concerned the smallest perturbation that makes a stable matrix unstable, and Demmel's concerned the smallest perturbations that makes two matrices with disjoint spectra have a common eigenvalue. We show that the truth of either of these conjectures would imply the truth of a third weaker and equally reasonable sounding conjecture. We then present a counterexample for this third conjecture which hence also serves as a counterexample to the first two.

A *stable matrix* is a matrix all of whose eigenvalues have negative real parts. Van Loan [Van Loan] recently made the following conjecture about the smallest perturbation of a stable matrix A which makes it unstable ($\|\cdot\|$ denotes the 2-norm):

Conjecture 1: Let A be stable. Let B be the closest unstable matrix to A, i.e. B is unstable and minimizes $\|A-C\|$ over all unstable C. Then B has an eigenvalue on the imaginary axis with the same imaginary part as some eigenvalue of A.

If this conjecture were true, it would lead to a simple computational scheme for computing $\|A-B\|$:

$$\|A-B\| = \min_{\lambda \in \sigma(A)} \sigma_{\min}(A - i \operatorname{Im}\lambda\, I)$$

where $\sigma(A)$ is the spectrum of A and $i=\sqrt{-1}$.

$\mathrm{sep}_\lambda(A,B)$ is the size of the smallest perturbations to A and B which makes them have a common eigenvalue:

$$\mathrm{sep}_\lambda(A,B) \equiv \min_\lambda \max(\sigma_{\min}(A-\lambda I)\ ,\ \sigma_{\min}(B-\lambda I)) \ .$$

Let $\sigma(A)$ denote the set of eigenvalues of A and similarly for $\sigma(B)$. Let $\mathrm{co}(X)$ denote the convex hull in the complex plane of the point set X. Demmel made the following conjecture about the minimizing λ in the definition of sep_λ:

Conjecture 2: The minimizing λ in the definition of sep_λ above lies in the convex hull $\mathrm{co}(\{\sigma(A)\ ,\ \sigma(B)\})$ of the spectra of A and B.

If this conjecture were true, it would greatly limit the region in the λ-plane that had to be searched for the minimizing λ.

In this short note we show if either of these two conjectures were true then a third weaker and equally reasonable sounding conjecture would be true. Then we will present a counterexample to this third conjecture. First we need some notation. Define $S(A,\epsilon)$ as the set of all eigenvalues of all matrices $A+\delta A$ for $\|\delta A\| \le \epsilon$:

$$S(A,\epsilon) = \{\lambda : \det(A+\delta A-\lambda I)=0 \ , \ \|\delta A\| \le \epsilon\} \ .$$

Conjecture 3: Let A be a matrix with a single eigenvalue λ. Then $S(A,\epsilon)$ is convex.

It is easy to see how Conjecture 3 is implied by either of the first two conjectures. First consider Conjecture 1. Note that $S(A,\epsilon)$ is connected, since any component must contain λ. As ϵ increases, $S(A,\epsilon)$ grows from a single point λ for $\epsilon=0$ to larger and larger sets. The value of ϵ for which this set first touches the imaginary axis is the size of the smallest perturbation that makes A unstable. Suppose there is a matrix A which violates conjecture 3. By multiplying A by a complex number ω of absolute value 1 and adding a multiple α of the identity, we can rotate and shift the eigenvalues of A so that A is stable and $S(A,\epsilon)$ makes any angle to the imaginary axis we want. Since $S(A,\epsilon)$ is nonconvex, we can clearly choose ω and α so that $S(\omega A+\alpha I,\epsilon)$ appears as in Figure 1. By varying ω and α slightly from these values, we can clearly make $S(\omega A+\alpha I,\epsilon)$ either first touch the imaginary axis only at a single point above the origin, or at a single point below the origin. Thus we can guarantee that it does not touch directly to the right of the single eigenvalue of $\omega A+\alpha I$. Thus Conjecture 1 is clearly violated. Therefore the truth of Conjecture 1 would imply the truth of Conjecture 3.

Now consider Conjecture 2. Again we proceed by contradiction. If Conjecture 3 were false, we could find an A with a single eigenvalue and an ϵ such that $S(A,\epsilon)$ were nonconvex. Again choose ω and α so that $S(\omega A+\alpha I,\epsilon)$ appears as in Figure 1, with the additional condition that the two points where $S(\omega A+\alpha I,\epsilon)$ contacts the imaginary axis are equidistant from the origin. Then $S(-\omega A - \alpha I,\epsilon)$ is clearly the reflection of $S(\omega A + \alpha I,\epsilon)$ in the origin, as shown in Figure 2. This violates Conjecture 2, since the convex hull of the spectrum of $\omega A + \alpha I$ and $-\omega A - \alpha I$ is a line segment through the origin passing between the single eigenvalue λ of $\omega A+\alpha I$ and $-\lambda$, and the minimizing λ in the definition of sep_λ must lie at one of the two points of contact on the imaginary axis. Therefore the truth of Conjecture 2 would imply the truth of Conjecture 3.

Finally, we present a counterexample to Conjecture 3, which is therefore also a counterexample to Conjectures 1 and 2. Let

$$A = \begin{bmatrix} -1 & -B & -B^2 \\ 0 & -1 & -B \\ 0 & 0 & -1 \end{bmatrix}$$

where $B >> 1$. A contour plot of $\log_{10}(\sigma_{\min}(A-\lambda I))$ in the λ plane (shapes of $S(A,\epsilon)$ for various ϵ) is shown in Figure 3 (for $B=100$); the nonconvexity of the contours is apparent. In fact, some of the $S(A,\epsilon)$ are not even simply connected! From Figure 3, we see that 0 is a local maximum of the function $\sigma_{\min}(A-\lambda)$. Thus, for example, $S(A,10^{-3})$ is essentially a disk with a small hole near the origin. In other words one can make $A+\delta A$ have any eigenvalue in an annulus about 0 with smaller $\|\delta A\|$ than is needed to make $A+\delta A$ have eigenvalue 0.

To see how much Conjecture 1 can be violated, consider the function $\sigma_{\min}(A-i\mu I)$, where μ is real. A plot of $\log_{10}(\sigma_{\min}(A-i\mu I))$ versus μ is shown in Figure 4 for $B=100$. We will show that for $\mu=2^{-1/2}$ $\sigma_{\min}(A-i\mu I)$ is at most $3^{3/2} / (2B^2)$ whereas $\sigma_{\min}(A)$ is of order $1/B$, which is much larger. Therefore $S(A,\epsilon)$ would touch the imaginary axis at about $\pm i2^{-1/2}$ for $\epsilon = O(B^{-2})$ but not the origin until $\epsilon \approx B^{-1}$.

The proof is a simple computation.

$$\sigma_{\min}(A-\lambda I) = \|(A-\lambda I)^{-1}\|^{-1} = \left\| \begin{bmatrix} \frac{-1}{1+\lambda} & \frac{B}{(1+\lambda)^2} & \frac{\lambda B^2}{(1+\lambda)^3} \\ 0 & \frac{-1}{1+\lambda} & \frac{B}{(1+\lambda)^2} \\ 0 & 0 & \frac{-1}{1+\lambda} \end{bmatrix} \right\|^{-1} .$$

When $\lambda = i\mu = 0$,

$$\sigma_{\min}(A - i\mu I) = \|A^{-1}\|^{-1} = \left\| \begin{bmatrix} -1 & B & 0 \\ 0 & -1 & B \\ 0 & 0 & -1 \end{bmatrix} \right\|^{-1} \approx \frac{1}{B}$$

for $B >> 1$. When $\lambda = i\mu \neq 0$,

$$\sigma_{\min}(A - i\mu I) = \|(A - i\mu I)^{-1}\|^{-1} \leq \left|\frac{-i\mu B^2}{(1+i\mu)^3}\right|^{-1} = \frac{(1+\mu^2)^{3/2}}{|\mu| B^2}$$

which as a function of μ reaches its minimum $3^{3/2} / (2\, B^2)$ at $\mu = 2^{-1/2}$.

If we let A be n by n and of the same structure as before:

$$A = \begin{bmatrix} -1 & -B & \cdot & \cdot & -B^{n-1} \\ & -1 & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & -1 & -B \\ & & & & -1 \end{bmatrix}$$

then $\sigma_{\min}(A) \approx B^{-1}$ as before and $\sigma_{\min}(A - i\mu I)$ achieves its minimum $O(B^{1-n})$ for $\mu = O(1)$. Thus for large B and/or large n, using Conjectures 1 and 2 as computational heuristics can lead to very bad results.

Note however that the matrix A is quite special: not only is it defective but it is nearly derogatory. Of course perturbing A slightly would yield a matrix with distinct eigenvalues with similarly shaped $S(A, \epsilon)$, so defectiveness per se is not essential, but nearness to a defective and derogatory matrix. It appears that if A is far from a derogatory matrix (i.e. A can be block diagonalized with one block per eigenvalue using a well-conditioned similarity), then one cannot go too far wrong using Conjecture 1 as a heuristic, and since derogatory matrices are quite rare (in the sense that a random matrix is unlikely to be very close to one [Demmel]), the heuristic is likely to be reliable.

References

[Demmel] J. Demmel, "A Numerical Analyst's Jordan Form," Dissertation, May 1983, Computer Science Dept., University of California, Berkeley

[Van Loan] C. Van Loan, "How Near is a Stable Matrix to an Unstable Matrix?" in *Linear Algebra and its Role in Systems Theory,* vol. 47 of Contemporary Mathematics, American Mathematical Society, 1985

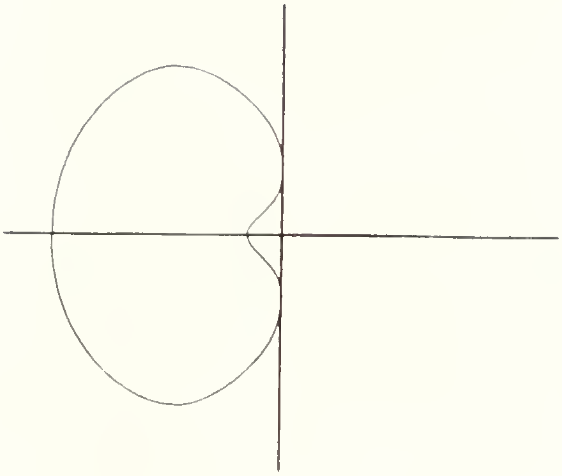

Figure 1.

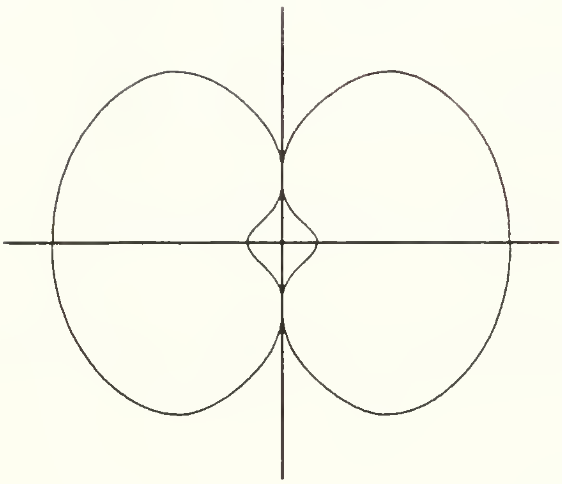

Figure 2.

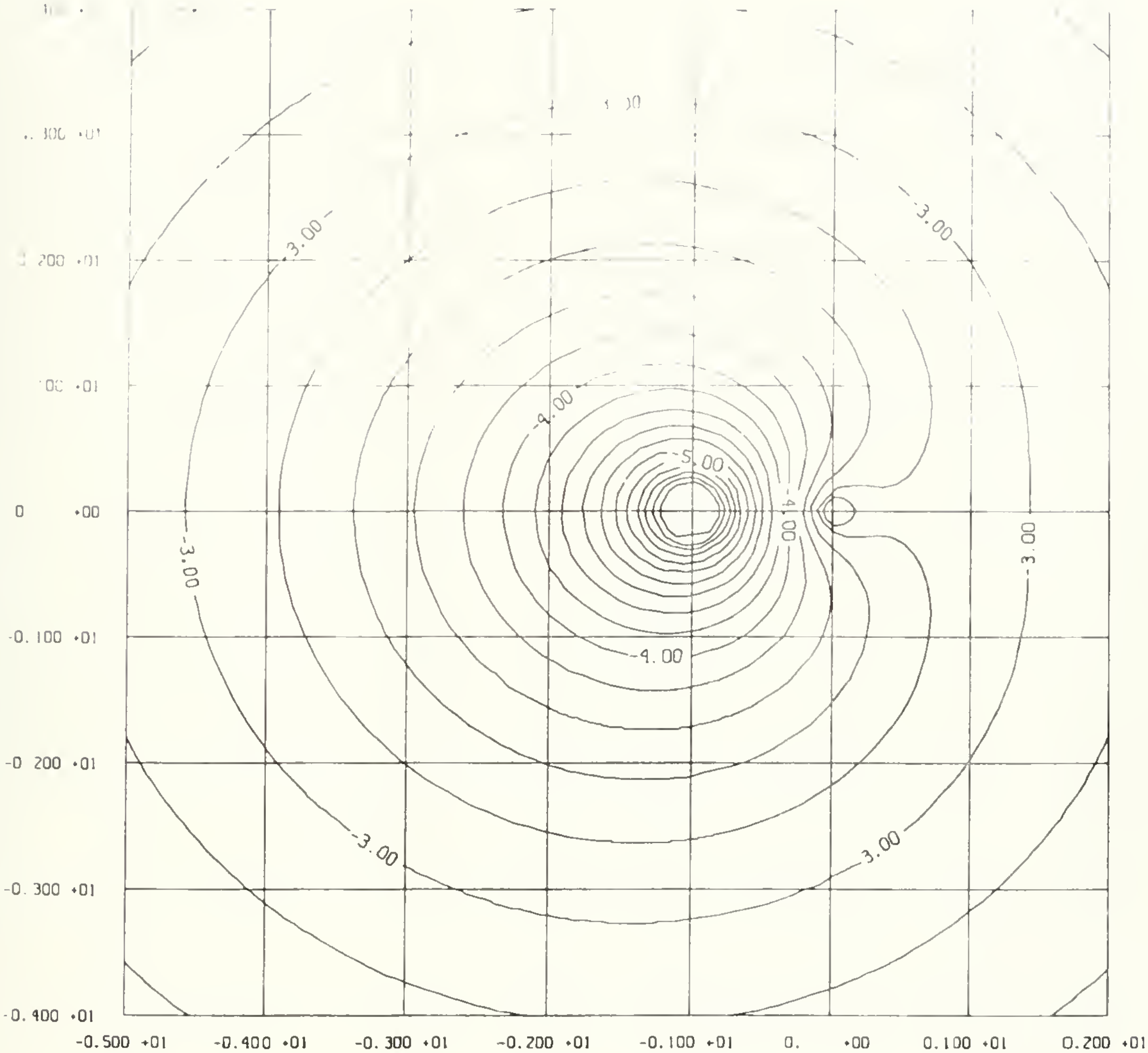

Figure 3. Contour plot of $\log_{10}(\sigma_{\min}(A - \lambda I))$ in the λ-plane

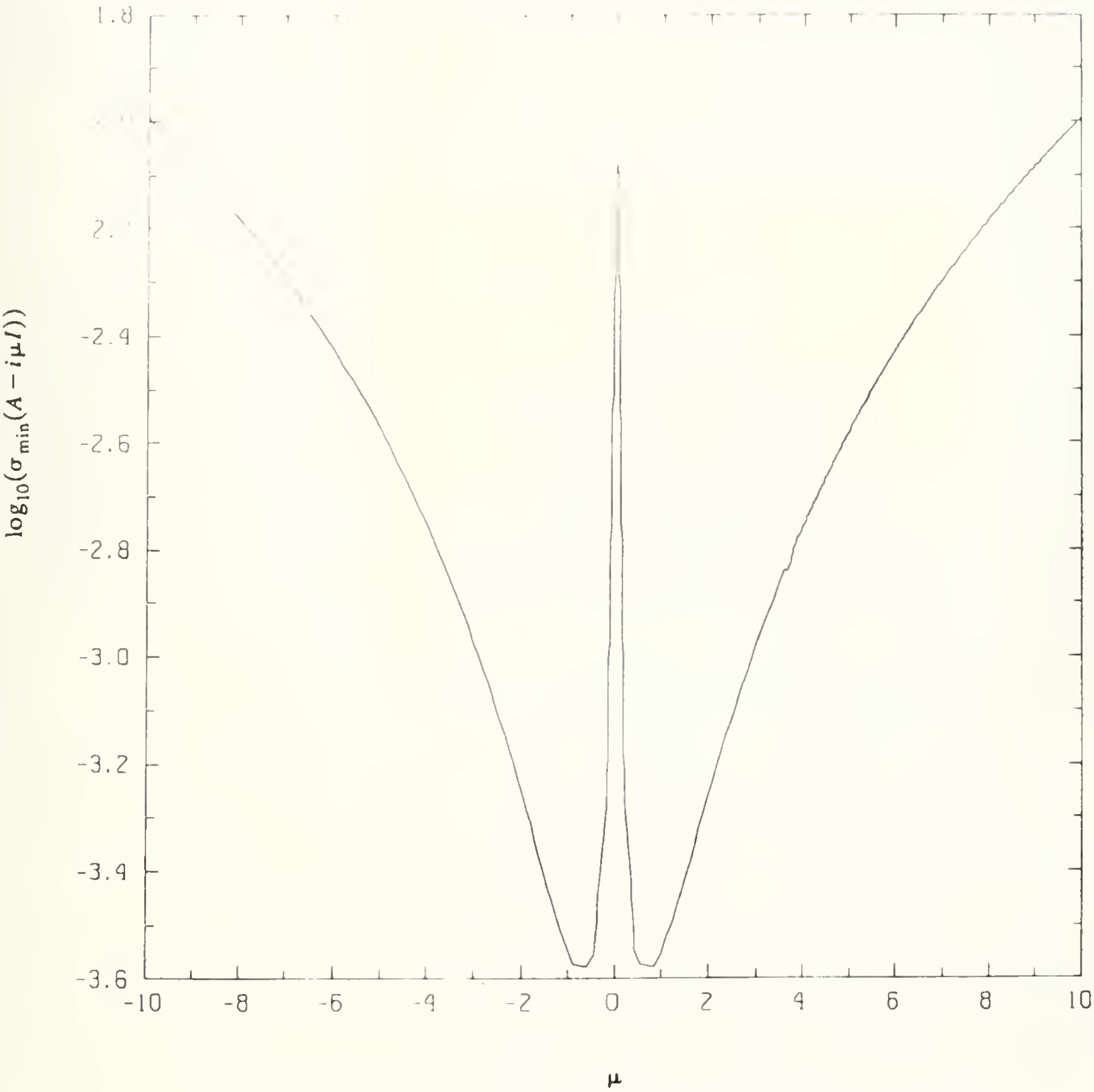

Figure 4. Graph of $\log_{10}(\sigma_{\min}(A - i\mu I))$ versus μ

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